As well as adding, subtracting and multiplying complex numbers, you can also divide two complex numbers by each other, i.e. $\mathbb{C}$ satisfies the axioms of a *field*: Precisely, let $\boldsymbol{z}$, w, $s \in \mathbb{C}$. Then we have the following properties:

Field axioms:

the addition axioms, which correspond to vector space axioms in $\mathbb{R}^2$:

$$\boldsymbol{z} + w = w + \boldsymbol{z}$$

$$\boldsymbol{z} + (w + s) = (\boldsymbol{z} + w) + s$$

$$\boldsymbol{z} + 0 = \boldsymbol{z}$$

$$\boldsymbol{z} + (-1 \cdot \boldsymbol{z}) = 0;$$

the multiplication axioms, which are straightforward and which you will check in homework:

$$\boldsymbol{z}\, w = w\, \boldsymbol{z}$$

$$(\boldsymbol{z}\, w)\, s = \boldsymbol{z}(w\, s)$$

$$1\, \boldsymbol{z} = \boldsymbol{z}\ ;$$

distributive property of multiplication over addition.

$$\boldsymbol{z}(w + s) = \boldsymbol{z}\, w + \boldsymbol{z}\, s;$$

and finally

each $\boldsymbol{z} \neq 0$ has unique $\boldsymbol{z}^{-1} \in \mathbb{C}$ which we write as $\frac{1}{\boldsymbol{z}}$, such that $\boldsymbol{z}\,\boldsymbol{z}^{-1} = 1$

and $\frac{w}{\boldsymbol{z}} := w\,\boldsymbol{z}^{-1}$ (in particular $\frac{\boldsymbol{z}}{\boldsymbol{z}} = 1$).

Wed start.

Example 2 Verify that

$$(2+3i)^{-1} = \frac{1}{2+3\,i} = \frac{1}{13}(2-3\,i) = \frac{2}{13} - \frac{3}{13}i$$

How did I know that???

check $(2+3i)\frac{1}{13}(2-3i) = \frac{1}{13}(13+0i) = 1$ ✓.

Complex conjugation

Let $z = x + i\,y$ with $x, y \in \mathbb{R}$. Then the *complex conjugate* of z, also called z *bar* and written as $\overline{z}$ is defined to be

$$\overline{z} := x - i\,y$$

And the *modulus* or *absolute value* of z is defined to be

$$|\,z\,| := \sqrt{x^2 + y^2}$$

Check:

a) $\overline{zw} = \overline{z}\,\overline{w}$.

$z = x+iy$
$w = u+iv$

$\overline{(x+iy)(u+iv)} \overset{?}{=} \overline{z}\,\overline{w} = (x-iy)(u-iv)$

$\overline{(xu-yv)+i(xv+yu)} = (xu-yv) - i(xv+yu)$ ✓

b) $|\,z\,|^2 = z\,\overline{z}$ so $|\,z\,| = \sqrt{z\,\overline{z}}$. ✓

$(x+iy)(x-iy) = x^2+y^2$ ✓

and so the absolute value of a product is the product of the absolute values:

c) $|\,z\,w\,| = |\,z\,|\,|w|$

$|zw|^2 \underset{(b)}{=} zw\,\overline{zw} \underset{(a)}{=} zw\,\overline{z}\,\overline{w} = z\overline{z}\,w\overline{w} = |z|^2|w|^2$ ✓

and this is how you compute reciprocols:

$$\frac{1}{z} = \frac{1}{z}\,\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|\,z\,|^2}$$

$$\frac{1}{2+3i}\,\frac{2-3i}{2-3i} = \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i$$

$$\mathbb{C} := \{x + i\,y \mid x, y \in \mathbb{R}\}.$$
$$(x_1 + i\,y_1) + (x_2 + i\,y_2) := (x_1 + x_2) + i(y_1 + y_2)$$
$$(x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2),$$
$$\text{for all } x_1, y_1, x_2, y_2 \in \mathbb{R}.$$

Under the identification of $\mathbb{C}$ with $\mathbb{R}^2$, the definition for complex number addition just corresponds to vector addition in $\mathbb{R}^2$ (considered as a vector space), which we understand and as we illustrated in the previous example. The product of a real number with a complex number corresponds to scalar multiplication in $\mathbb{R}^2$, which we also understand geometrically.

$$\begin{aligned} \mathbb{C}: &\quad (x_1 + i\,y_1) + (x_2 + i\,y_2) := (x_1 + x_2) + i(y_1 + y_2) \\ \mathbb{R}^2: &\quad (x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2) \end{aligned}$$

$$\begin{aligned} \mathbb{C}: &\quad x_1(x_2 + i\,y_2) := x_1x_2 + i\,x_1\,y_2 \\ \mathbb{R}^2: &\quad x_1(x_2, y_2) := (x_1\,x_2, x_1\,y_2) \end{aligned}$$

The more general formula for complex multiplication has geometric meaning. This magic meaning is not immediately apparent using Cartesian coordinates, as the formula in $\mathbb{R}^2$ looks sort of mysterious. But polar coordinates will solve the mystery.

$$\begin{aligned} \mathbb{C}: &\quad (x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \\ \mathbb{R}^2: &\quad (x_1, y_1)(x_2, y_2) := (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)\,. \end{aligned}$$

Polar form of complex numbers and the geometric meaning of complex multiplication.

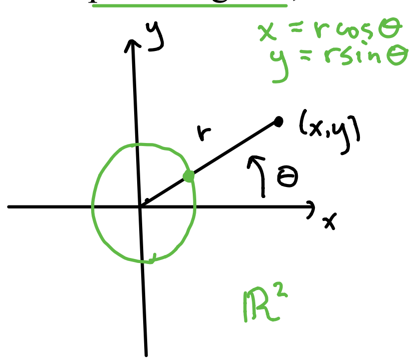
Recall polar coordinates in $\mathbb{R}^2$: Every non-zero vector in $\mathbb{R}^2$ can be expressed as

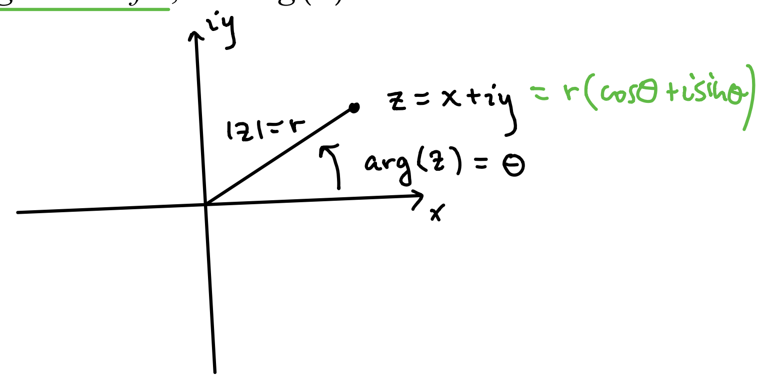
$$(x, y) = \left(r\cos\theta\,,\ r\sin\theta\right) = r\left(\cos\theta,\ \sin\theta\right)$$

where $r = \sqrt{x^2 + y^2}$ and θ is the angle in radians from the positive $x-$axis to the point (x, y), determined up to an integer multiple of 2π. In complex form this reads

$$z = x + i\,y = r\left(\cos\theta + i\sin\theta\right).$$

Note that $r = |\,z\,|$ is the absolute value of z, using complex notation. And we also have a special name for the polar angle θ, we call it the *argument of* z, or $arg(z)$ for short.

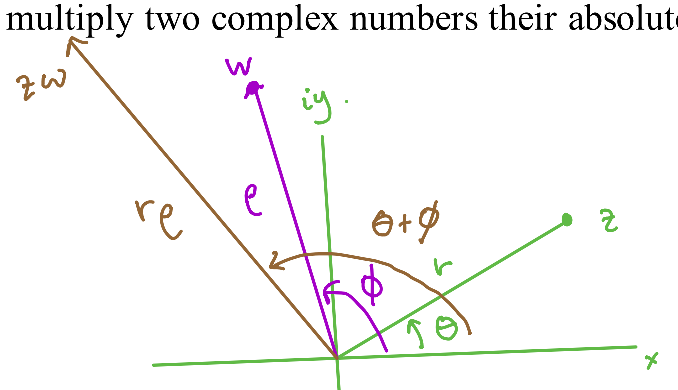




Theorem: Let $z = r\left(\cos\theta + i\sin\theta\right)$ and $w = \rho\left(\cos\phi + i\sin\phi\right)$ be complex numbers written in polar form. Then

$$z\,w = r\,\rho\,\left(\cos(\theta+\phi) + i\sin(\theta+\phi)\right).$$

In other words, when you multiply two complex numbers their absolute values multiply and their arguments add!



Pf $\quad zw = r(\cos\theta + i\sin\theta)\,\rho(\cos\phi + i\sin\phi)$

$$= r\rho\left[\underbrace{(\cos\theta\cos\phi - \sin\theta\sin\phi)}_{\cos(\theta+\phi)} + i\underbrace{(\cos\theta\sin\phi + \sin\theta\cos\phi)}_{\sin(\theta+\phi)}\right]$$

Note: If you use Euler's formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ from Math 2280 ,then the multiplication formula from the previous page is particularly nice and concise: Let

$$z = |z|(\cos(\theta) + i\sin(\theta)) = |z|\, e^{i\theta}$$ $\theta = \arg z$

$$w = |w|(\cos(\phi + i\sin(\phi)) = |w|\, e^{i\phi},$$ $\phi = \arg w$

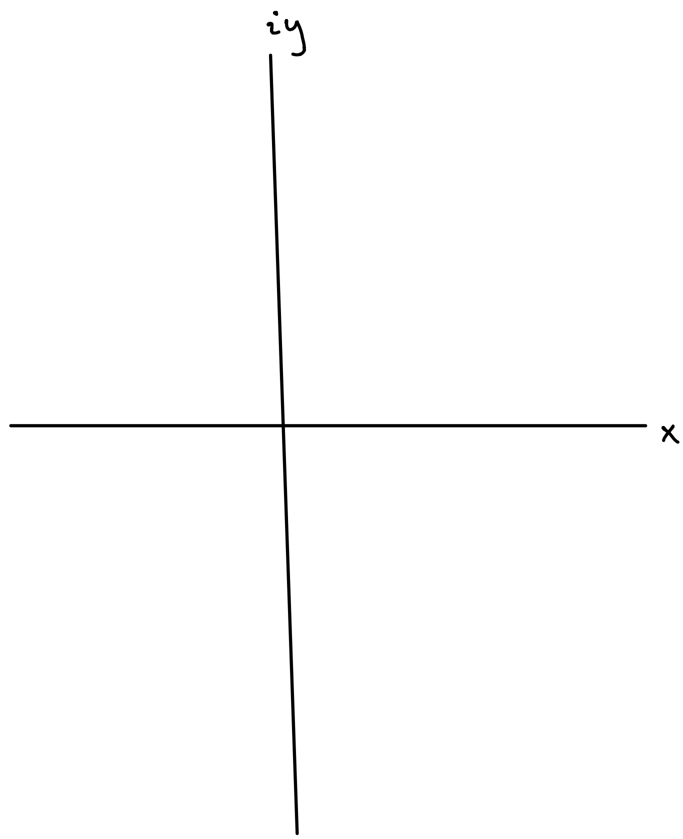
then product

previous page

$$z\,w = |z|\, e^{i\theta}\, |w|\, e^{i\phi} = |z|\,|w|\, e^{i\cdot(\theta+\phi)}$$

Example 4 Express $z = 1 + i$ in polar form. Compute $z^2, z^3, \dfrac{1}{z}$ using rectangular and polar form. Sketch!! To be continued!

skip!



Math 4200
Wednesday August 26

1.1-1.2 Algebra and geometry of complex arithmetic, continued.
We'll pick up in Monday's notes where we left off (there was still a lot to talk about there), and used today's notes to talk about solutions to polynomial equations.

Announcements: We'll try group quizzes at the end of class today!

Warm-up example: Use rectangular coordinates to find all complex solutions to the following. Sketch the solutions in the complex plane.

a) $z^2 = -9$

b) $z^3 = 1$.

a) $z^2 = -9$

$z = \pm 3i$ just used old alg.

or. let

$z = x + iy$

$(x+iy)^2 = -9$

$(x^2 - y^2) + 2ixy = -9 + 0i$

$2xy = 0$

Im $x = 0$ or $y = 0$

Re $-y^2 = -9$ $\cancel{x^2 = -9}$

$y = \pm 3$ $\emptyset$.

$0 \pm 3i$

b) $z^3 - 1 = 0$

$(z-1)(z^2+z+1)$

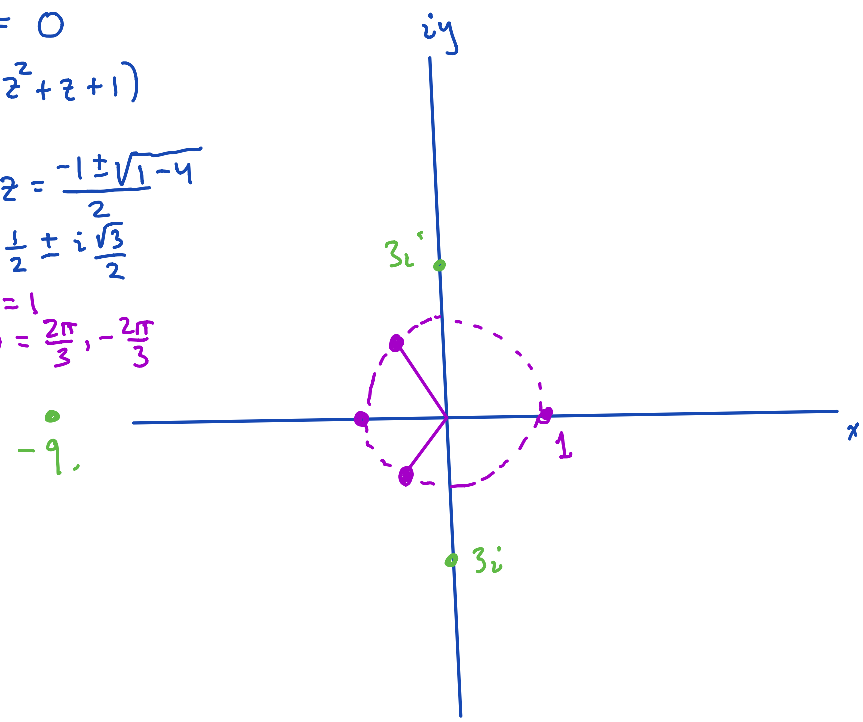
$z = 1$

Q.F. : $z = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$r = 1$

$\theta = \frac{2\pi}{3}, -\frac{2\pi}{3}$



After we discuss the polar form of complex numbers, we'll come back re-solve the page 1 equations that way.

a) $z^2 = -9$
b) $z^3 = 1$.

a) $z = re^{i\theta}$, $-9 = 9e^{i\pi}$

$z^2 = -9$

$(re^{i\theta})^2 = 9e^{i\pi}$

$\parallel$

$r^2 e^{i(2\theta)} = 9e^{i\pi}$

modulus: $r^2 = 9 \Rightarrow r = 3$

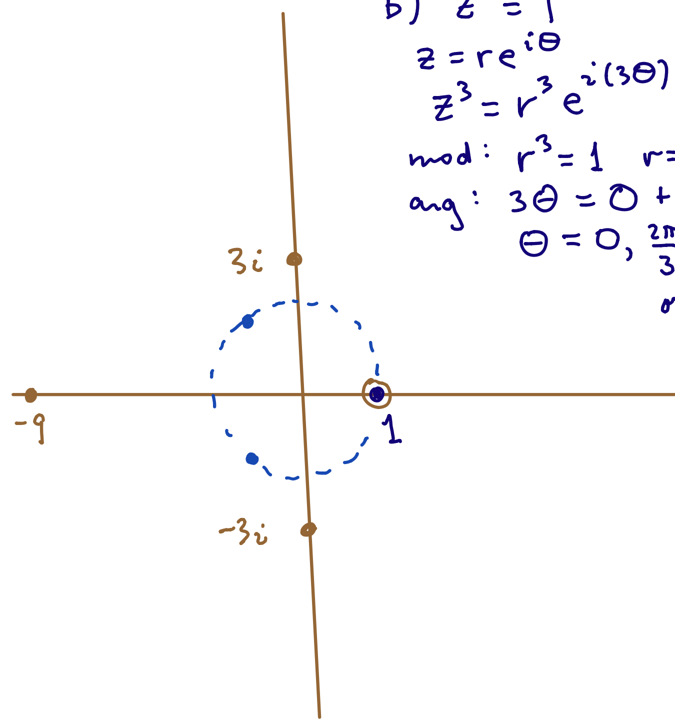
ang: $2\theta = \pi + 2\pi k, \ k \in \mathbb{Z}$

$\div 2$ $\quad \theta = \frac{\pi}{2} + \pi k$

$\theta = \frac{\pi}{2}, \frac{3}{2}\pi$ or equiv.

$z = 3e^{i\pi/2} = 3i$

or $z = 3e^{i(\frac{3}{2}\pi)} = -3i$



b) $z^3 = 1$

$z = re^{i\theta}$

$z^3 = r^3 e^{i(3\theta)} = 1e^{i0}$

mod: $r^3 = 1 \quad r = 1$

ang: $3\theta = 0 + 2\pi k$

$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

or equiv.